Noncommutative Geometry and the Spectral Action, an Overview.

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Introduction

The Basic Idea of NCG

The goal - to construct a geometric approach to including noncommutativity (as seen in quantum physics, QFT, etc.) into a given physical theory based on geometry (e.g. Einstein Gravity) .

A fundamental motivation for noncommuting coordinates is the DFR paradox - by marrying only the essential assumptions of QFT and GR, we get relations of the type:

$$[x^\mu, x^\nu] = i\theta^{\mu\nu} \qquad\qquad \Delta x^\mu \Delta x^\nu \geq ||\theta^{\mu\nu}||$$

Some examples where this might be used:

- Effective models, e.g. electrons in a 2-dimensional material through the Moyal approach: $$[x^\mu, x^\nu] = i\theta^{\mu\nu}$$
- Quantizing a theory via *-products: $$f \star g = fg + \sum_{n=1}^{\infty} \theta^n \, C_n(f, g)$$
- Constructing theories (such as the Standard Model) on curved backgrounds (on equal footing with gravity)
- Constructing new theories of say quantum gravity

The Road to Noncommutative Geometry

We usually think of the world around us through observables, i.e. functions on a manifold and it’s taken for granted that we can use these observables to obtain all the information of the space we’re living in.

This seems to imply that, knowing the physical laws, there exists some direct correspondence between the algebra of functions on a manifold and the topological and differentiable structure of the manifold itself.

This kind of thought process was formalized by Connes in the 1980s into what is called the Spectral Triple:

$$(\mathcal{A}, \mathcal{H}, \mathcal{D})$$

which provides a piece by piece, mathematically formal relation between a Riemannian manifold structure and an abstract algebra.

The Spectral Triple

$\mathcal{A}$ - an abstract algebra, corresponding to the algebra of functions on the manifold

$\mathcal{H}$ - a Hilbert space on which the above algebra is faithfully represented

$\mathcal{D}$ - the Dirac operator, representing the analogue of the Laplace operator, which determines the geometrical structure on the manifold

The Topological Equivalence

At this level, we only need the algebra and the Hilbert space $(\mathcal{A}, \mathcal{H})$ to guarantee the correspondence by means of two theorems:

Theorem 1. ***(Gelfand-Naimark-Segal; The Gelfand duality)*** *Every abstract C^*-algebra $\mathcal{A}$ is isometrically $*$-isomorphic to a concrete C^*-algebra of operators on a Hilbert space H. If the algebra $\mathcal{A}$ is separable then we can take H to be separable.*

Theorem 2. ***(Gelfand-Naimark)*** *If a C^*-algebra is commutative then it is an algebra of continuous functions on some (locally compact, Hausdorff) topological space.*

However, one should note that if $\mathcal{A}$ was to be made noncommutative, as an arbitrary algebra can be, it would become unclear what “noncommutative topological space” this would correspond to.

Differentiable Structure

A construction of vector fields algebraically comes down to noticing what the vector fields look like on manifolds and requiring an analogous structure on the algebra:

Linearity & Leibniz Rule

As the space of derivations of an algebra satisfy exactly these rules, they can be taken to be equivalent to vector fields.

Forms are then obtained as the graded differential algebra of duals to vector fields.

Geometry

This is where the Dirac operator $\mathcal{D}$ first comes in. Taking it to be self-adjoint as well as having a compact resolvent (i.e. the eigenvalues behave in some nice way) implies a lot of properties, among other things:

$$\|[\mathcal{D}, f]\| \leq \infty \quad \forall f \in \mathcal{A}$$

It can be proven, as a result of Kantorovich transportation theory, that the analogue of the manifold distance is then:

$$d(a, b) = \sup\{|f(a) - f(b)| \colon [\mathcal{D}, f] \leq 1\}$$

This makes sense as the Dirac operator is constructed as the square root of the Laplace which is essentially the metric tensor. Analogously to the standard Klein-Gordon vs Dirac equation thing, it seems plausible that the Dirac is then more general than the Laplace / metric tensor.

The Topology-Algebra Dictionary

In summary, this comes down to the following correspondence between the **topological** and the **algebraic:**

Points on Topological Space	(pure) States on the Algebra
$C^\infty(\mathcal{M})$	Noncommutative Algebra
Vector Fields	Derivations of Algebra
deRham Complex	Derivation Based Calculus
Vector Bundles of Spaces	Modules over Algebras
de Rham Cohomology	Cyclic Homology
Riemann Manifold	Spectral Triple (A,H,D)
Riemann Metric	Spectral Distance (Dirac op.)
Atiyah-Singer theorem	Connes-Moscovici Index Thm.

Given a Riemannian manifold $(\mathcal{M}, g)$ one can prescribe a "**canonical Spectral Triple**" $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ as follows:

- Take as the algebra $\mathcal{A}$ the algebra of C^∞-functions on the manifold $\mathcal{M}$
- The Hilbert space $\mathcal{H}$ is the space of square integrable sections of the spinor bundle on $\mathcal{M}$ - $L^2(S)$
- The Dirac operator $\mathcal{D}_M$ as the Levi-Civita connection lifted to the spinor bundle over $\mathcal{M}$.

Such a spectral triple:

$$(C^\infty(M), L^2(S), \mathcal{D}_M)$$

then contains all the same information as a Riemannian manifold $(\mathcal{M}, g)$.

Next we might ask if we can go the other way; is it the case that given a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ we can find a Riemannian manifold it corresponds to.

The Connes Reconstruction Theorem

In recent years (2004.) Connes published the paper “On The Spectral Characterization of Manifolds”, proving the equivalence (if and only if) between compact oriented smooth Riemannian manifolds and commutative spectral triples which generalizes the Gelfand-Naimark theorems from earlier.

The idea of talking about noncommutative spaces relies on the existence of the above equivalence - Riemannian manifolds **are** equivalent to commutative spectral triples. Considering this, one could take a noncommutative spectral triple and claim that it is the correct extension of a Riemannian manifold to a “noncommutative” one. Doing calculations on such a spectral triple would then correspond to doing calculations on a “noncommutative manifold”.

From the Spectral Triple to the Spectral Action

From the Spectral Triple to the Action

Obtaining physical laws usually stems from an action principle loosely constructed as follows:

- Identify the symmetries of the physics
- Construct all scalar quantities that respect said symmetries
- Add them all up and call this the action
- Apply the variational principle to the action and obtain equations of motion

Now, of course what one has at hand when working with Spectral Triples is a Hilbert space (on which the algebra is represented) with its inner product and the Dirac operator on said Hilbert space and it remains to be seen what scalars can be made using this construction.

The Spectral Action

The Spectral action, as prescribed by Connes then takes the form:

$$S_{\text{Spectral}} = < \psi, D\psi > + \mathrm{Tr}\left(\chi\left(\frac{D}{\lambda}\right)\right)$$

Almost commutative Spectral Triples and the Standard Model

The Standard Model + GR

The way to proceed is to construct what is known as an almost-commutative spectral triple, taking the algebra:

$$A = C^\infty(M) \otimes A_F$$

We want the symmetry group:

$$U(1) \times SU(2) \times SU(3) \rtimes \mathrm{Diff}(M)$$

as this is exactly the standard model..

The finite space A_F thus has to be picked to reproduce the correct gauge symmetries of the standard model and the choice turns out to be:

$$A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$$

The Spectral Action with the product Dirac:

$$D = \partial\!\!\!/_M \otimes 1 + \gamma_5 \otimes D_F$$

then produces two parts:

- The fermionic Dirac term $< \psi, D\psi >$ which includes fermion kinetic terms, Yukawa couplings and fermion-gauge boson interactions
- The bosonic action:

$$\mathrm{Tr}\left(\chi\left(\frac{D}{\lambda}\right)\right)$$

containing everything from the Einstein-Hilbert action with cosmological constant, Weyl Gravity, the bosonic Standard Model action and some additional terms among which is the Higgs, appearing on its own, without being put in by hand.

$$
\begin{aligned}
S_{\text{Spectral}} = & \frac{45\lambda^4}{4\pi^2} f_0 \int d^4x \sqrt{g} \\
& + \frac{3\lambda^2}{4\pi^2} f_2 \int d^4x \sqrt{g} \left[\frac{5}{4} R - 2y^2 H^* H \right] \\
& + \frac{f_4}{4\pi^2} \int d^4x \sqrt{g} \left[\frac{5}{160} \left(12 R^{\mu}_{;\mu} + 11 R^* R^* \right.\right. \\
& \left. - 18 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \right) \\
& + 3y^2 \left(D_\mu H^* D^\mu H - \frac{1}{6} R H^* H \right) \\
& + g_{03}^2 G^i_{\mu\nu} G^{\mu\nu i} + g_{02}^2 F^\alpha_{\mu\nu} F^{\mu\nu\alpha} \\
& + \frac{5}{3} g_{01}^2 B_{\mu\nu} B^{\mu\nu} \\
& \left. + 3z^2 (H^* H)^2 - y^2 (H^* H)^{\mu}_{;\mu} \right] + O\left(\frac{1}{\lambda^2} \right)
\end{aligned}
$$

$$
\frac{g_{03}^2 f_4}{\pi^2} = 1
$$

$$
g_{03}^2 = g_{02}^2 = \frac{5}{3} g_{01}^2
$$

The Normalized Action:

$$\int d^4x\,\sqrt{g}\,\Bigg[\frac{1}{2\kappa_0^2}\,R - \mu_0^2(H^*H) + a_0\,C_{\mu\nu\rho\sigma}\,C^{\mu\nu\rho\sigma}$$

$$+\, b_0\,R^2 + c_0\,{}^*R^*R + d_0\,R_{;\mu}{}^{\mu}$$

$$+\, e_0 + \frac{1}{4}\,G^i_{\mu\nu}\,G^{\mu\nu i} + \frac{1}{4}\,F^{\alpha}_{\mu\nu}\,F^{\mu\nu\alpha}$$

$$+\, \frac{1}{4}\,B_{\mu\nu}\,B^{\mu\nu} + |D_\mu\,H|^2 - \xi_0\,R|H|^2 + \lambda_0(H^*H)^2\Bigg]$$

Thank you!